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PORTFOLIO EQUILIBRIUM BETWEEN LAND, GOLD, AND CAPITAL IN A GROWTH MODEL WITH AGRICULTURAL, INDUSTRIAL, AND ENVIRONMENTAL SECTORS

Wei-Bin Zhang
School of International Management, Ritsumeikan Asia University, Beppu-City, Japan
E-mail: wbz1@apu.ac.jp

Abstract
This study builds an endogenous growth model of capital accumulation and environmental change with portfolio equilibrium between land, gold, and physical wealth. The model is based on microeconomic mechanisms for determining the prices, rents and distributions of land, gold, and physical wealth. The economy consists of industrial, agricultural, and environmental sectors. The land is distributed between agricultural production and residential use. Gold is used for decoration and owned for storing value. The model is an integration of the neoclassical growth theory and Ricardian theory. The household's decision is modeled with an alternative approach proposed by Zhang (1993). We simulate the motion of the economy. We demonstrate that the economic system has a unique stable steady state. We also conduct comparative dynamic analysis to demonstrate how exogenous changes in the propensity to use gold, the propensity to consume housing, the propensity to consume industrial goods, the propensity to consume agricultural goods, the propensity to save, the environment sector's productivity, the tax rate on consumption of industrial goods, the population, and the tax rate on the land rent income affect the economic system during transitory processes and in long-term equilibrium.

Keywords: Land value, Gold value, Environment, Taxes, Wealth accumulation

JEL classification: O41, O44, Q11, Q56

1. Introduction

Modern economics are characterized of multiple assets. People store their wealth in different assets such as land, housing, silver, gold, stocks, and money. As economic growth brings about changes in wealth and the change in wealth will cause changes in portfolio equilibrium, it is necessary to develop a growth theory with portfolio choice. Nevertheless, modern economic theory still lacks dynamic economic growth models with multiple assets. The purpose of this study is to introduce land and gold into a neoclassical growth model with endogenous capital accumulation and environmental change.

Land prices vary over time and space in association with economic development. Nevertheless, only a few theoretical growth models include endogenous determination of land value in the economic growth theories with microeconomic foundation. Since Ricardo published his well-known *On the Principles of Political Economy and Taxation* published in 1817, the determination of land rent is a main concern of theoretical land economics. The Ricardian system attempts to interconnect wages, interest rate, and rent together in a compact theory. Ricardo (1821: preface) pointed out: "The produce … is divided among three classes of the commodity, namely, the proprietor of land, the owners of the stock or capital necessary for its cultivation, and laborers by whose industry it is cultivated. But in different stages of the society, the proportions of the whole produce of the earth which will be allotted to each of these classes, under the names of rent, profits, and wages, will be essentially different; depending mainly on the actual fertility of the soil, on the accumulation of capital and population, and on the skill, ingenuity, and the instruments in agriculture." Nevertheless, a few formal economic models can properly deal with Ricardo's ambition in a compact framework on microeconomic foundation. Since the publication of the *Principles*, many attempts have been done to extend or generalize Ricardo's system (Barkai, 1959, 1966; Pasinetti, 1960, 1974; Brems, 1970; Caravale and Tosato, 1980; Casarosa, 1985; Negish, 1989; Morishima, 1989). Nevertheless, we can still apply what Ricardo (1821: preface) observed long time ago to describe the current situation: "To determine the laws which regulate this distribution, is the principal problem in Political Economy: much as the science has been improved by the writings of Turgot, Stuart, Smith, Say, Sismondi, and others, they afford very little satisfactory information respecting the natural course of rent, profit, and wages." In Ricardo's statement there is no reference to land value (price).. As Cho (1996: 145) stated, "During the past decade, the number of studies on intertemporal changes in house prices has increased rapidly because of wider availability of extensive micro-level data sets, improvements in modeling techniques, and expanded business applications." The literature on house and land prices has been increasingly expanding since then (e.g., Bryan and Colwell, 1982; Case and Quigley, 1991; Calhoun, 1995; Capozza and Seguin, 1996; Alpanda, 2012; Alexander, 2013; Du and Peiser, 2014; Kok *et al.* 2014). According to Liu *et al.* (2011: 1), "Although it is widely accepted that house prices could have an important influence on macroeconomic fluctuations, quantitative studies in a general equilibrium framework have been scant." Since land value is related to physical wealth which can be accumulated through saving, we need microeconomic mechanism to determine land value and saving in an integrated framework.

This study integrates not only the basic economic mechanisms of the Ricardian and neoclassical growth theories, but also introduces gold price dynamics to the growth theory with agricultural and industrial sectors. Dynamics of gold prices are not properly examined in the literature of economic dynamics with capital accumulation (Barro, 1979; Bordo and Ellson, 1985; Dowd and Sampson, 1993; Chappell and Dowd, 1997). In addition to introducing gold into the growth theory, this study takes account of environmental change. Environment affects productivities of firms and welfare of households. We study a dynamic interdependence between consumption, production, and government's environmental policies. The dynamic relations between consumption and pollution have been extensively analyzed in the literature of formal economic analysis since the publication of the seminal papers by Ploude (1972) and Forster (1973) (e.g., Pearson, 1994; Grossman, 1995; Copeland and Taylor, 2004; Dinda, 2004; Stern, 2004; Dasgupta *et al.*, 2006; Brock and Taylor, 2006; Managi, 2007; Kijima *et al.*, 2010; and Tsurumi and Managi, 2010).

The unique feature of this paper is to connect land and gold value determination with capital accumulation and environmental change. We model the economic aspects on the basis of the neoclassical growth theory. The Solow model is the key model in the neoclassical growth theory. The theory is mainly concerned with endogenous capital accumulation (Solow, 1956; Burmeister and Dobell, 1970; Azariadis, 1993; Barro and Sala-i-Martin, 1995). The main framework of this study is based on the neoclassical growth theory. We follow the Solow model in modeling economic production. Nevertheless, we analyze household behavior by the approach proposed by Zhang (1993). It should be noted that this study is a synthesis of Zhang's recent two models (Zhang, 2015, 2016). This paper is organized as follows. Section 2 develops the growth model with endogenous physical capital accumulation and environmental change. Section 3 examines dynamic properties of the model and simulates the model. Section 4 carries out comparative dynamic analysis. Section 5 concludes the study. The appendix proves the results in section 3.

2. The model

The model in this study integrates Zhang's two models. Zhang (2015) builds a model with endogenous environment, while Zhang (2016) constructs a growth model of portfolio equilibrium between land, gold, and physical wealth. This study deals with a dynamic economy with industrial, agricultural, and environment sectors with portfolio equilibrium between land, gold, and physical wealth. The industrial sector produces goods which is the same as that in Solow's one-sector neoclassical growth model. The commodity can be used both for investment and consumption. The agricultural good is used for consumption. The environmental sector keeps environment clean and is financially supported by the government. The government income comes for sources of taxation on producers and consumers. The economy has a constant and homogeneous population N and has a constant homogenous land L. The land is owned by households and is distributed between housing and agricultural production in the competitive land market. All the markets are perfectly competitive. This implies that the rate of interest $r(t)$ and wage rate $w(t)$ are determined by markets. We select industrial goods to serve as numeraire.

The industrial sector

Let $N_i(t)$ and $K_i(t)$ represent respectively labor force and physical capital employed by the industrial sector. The production function $F_i(t)$ which is neoclassical one and homogeneous of degree one with the inputs is specified as follows

$$F_i(t) = A_i \Gamma_i(E(t)) K_i^{\alpha_i}(t) N_i^{\beta_i}(t), \quad A_i, \alpha_i, \beta_i > 0, \quad \alpha_i + \beta_i = 1, \tag{1}$$

where $\Gamma_i(E)$ is a function of the environmental quality measured by the level of pollution, $E(t)$, and A_i, α_i, and β_i are parameters. The parameters α_i and β_i are the output elasticicties of capital and labor, respectively. The productivity of the sector is negatively related to the pollution level, i.e., $d\Gamma_i / E \le 0$. The marginal conditions are given by

$$r(t) + \delta_k = \frac{\alpha_i \bar{\tau}_i F_i(t)}{K_i(t)}, \quad w(t) = \frac{\beta_i \bar{\tau}_i F_i(t)}{N_i(t)}, \tag{2}$$

where δ_k is the fixed depreciation rate of physical capital and $\bar{\tau}_i \equiv 1-\tau_i$ where τ_i is the fixed tax rate on the industrial sector's output, $0<\tau_i<1$.

The agricultural sector

The inputs of agricultural production are capital $K_a(t)$, labor force $N_a(t)$, and land $L_a(t)$. The production function is taken as follows

$$F_a(t)=A_a\,\Gamma_a(E(t))K_a^{\alpha_a}(t)N_a^{\beta_a}(t)L_a^{\varsigma}(t),\ \ A_a,\alpha_a,\beta_a,\varsigma>0,\ \ \alpha_a+\beta_a+\varsigma=1, \tag{3}$$

where $\Gamma_a(E(t))$ is similarly defined as $\Gamma_i(E(t))$, and A_a, α_a, β_a, and ς are parameters. We use $p_a(t)$ and $R_L(t)$ to stand for, respectively, the price of agricultural goods and the land rent. The marginal conditions are

$$r(t)+\delta_k=\frac{\alpha_a\,\bar{\tau}_a\,p_a(t)F_a(t)}{K_a(t)},\ \ w(t)=\frac{\beta_a\,\bar{\tau}_a\,p_a(t)F_a(t)}{N_a(t)},\ \ R_L(t)=\frac{\varsigma\,\bar{\tau}_a\,p_a(t)F_a(t)}{L_a(t)}, \tag{4}$$

where $\bar{\tau}_a \equiv 1-\tau_a$ where τ_a is the fixed tax rate on the agricultural sector's output, $0<\tau_a<1$.

Portfolio equilibrium between physical wealth, gold and land

Let $p_L(t)$ and $p_G(t)$ stand for, respectively, the prices of land and gold. For convenience of analysis, it is assumed that gold can be "rented" through markets for decoration use. The rent of gold is denoted by $R_G(t)$. The gold owned by the representative household is assumed to be fully used either by the household for decoration or rented out to other households. Land may be owned by different agents under various institutions. We assume that land is privately owned by households. We assume that land can be sold and bought in free markets without any friction and transaction costs. Land use will not waste land and land cannot regenerate itself. Households own land and physical wealth. We use $p_L(t)$ to denote the price of land. Consider now an investor with one unity of money. He can either invest in capital good thereby earning a profit equal to the net own-rate of return $r(t)$ or invest in land (gold) thereby earning a profit equal to the net own-rate of return $R_L(t)/p_L(t)$ ($R_G(t)/p_G(t)$). As we assume capital and land markets to be at competitive equilibrium at any point in time, two options must yield equal returns, i.e.

$$\frac{(1-\tau_L)R_L(t)}{p_L(t)}=\frac{(1-\tau_G)R_G(t)}{p_G(t)}=(1-\tau_k)r(t), \tag{5}$$

where τ_L, τ_G and τ_k are the fixed tax rates on the land rent income, the gold rent income, and income from interest payment on wealth, respectively. This equation enables us to determine the choice between owning land, gold, and physical wealth. It is made under many strict conditions. For instance, we neglect any transaction costs and any time needed for buying and selling. Expectations on land and gold are complicated. Equation (5) also implies perfect information and rational expectation.

The current income and disposable income

For simplicity, we use the lot size to stand for housing. Consumers decide consumption levels of industrial and agricultural goods, gold, and lot size, as well as on how much to save. The land distribution between the agricultural use and residential use is determined in competitive land markets. The land rent is equal both for the land used for agricultural production and for residential use. This study uses the approach to consumers' behavior proposed by Zhang (1993). We denote physical wealth by $\bar{k}(t)$ and land $\bar{l}(t)$ owned by the representative household, respectively. Let $\bar{g}(t)$ stand for the amount of gold owned by the household. The total value of wealth owned by the household $a(t)$ is the sum of the three assets

$$a(t) = \bar{k}(t) + p_L(t)\bar{l}(t) + p_G(t)\bar{g}(t). \tag{6}$$

Per capita current income from the interest payment $r(t)\bar{k}(t)$, the wage payment $w(t)$, the land revenue $R_L(t)\bar{l}(t)$, and the gold revenue $R(t)\bar{l}(t)$, is given by

$$y(t) = (1 - \tau_k)r(t)\bar{k}(t) + (1 - \tau_w)w(t) + (1 - \tau_L)R_L(t)\bar{l}(t) + (1 - \tau_G)R_G(t)\bar{g}(t), \tag{7}$$

where τ_w is the fixed tax rate on the wage. It should be noted that in the Solow growth model $y(t)$ is called disposable income. In our approach $y(t)$ is called the current income. In our approach the disposable income is the sum of the current value and the value of the household's wealth. What a household is not only what the household currently earns, but also includes the value of the household's wealth (if the wealth could be sold instantaneously without any transaction costs). In this study we assume that agents can sell and buy their assets instantaneously without any transaction costs. The per capita disposable income is given by

$$\hat{y}(t) = y(t) + a(t). \tag{8}$$

The budget, utility function, and optimal decision

At each point in time, a consumer would distribute the disposable income between the saving $s(t)$, the consumption of industrial goods $c_i(t)$, the consumption of agricultural goods $c_a(t)$, gold use for decoration $\hat{g}(t)$, and the lot size $l_h(t)$. Let $\tilde{\tau}_c$, $\tilde{\tau}_a$, $\tilde{\tau}_h$, and $\tilde{\tau}_G$ stand for, respectively, the fixed tax rates on consumption of industrial good, agricultural good, housing, and use of gold. The budget constraint is given by

$$(1 + \tilde{\tau}_c)c_i(t) + s(t) + (1 + \tilde{\tau}_a)p_a(t)c_a(t) + (1 + \tilde{\tau}_h)R_L(t)l_h(t) + (1 + \tilde{\tau}_G)R_G(t)\hat{g}_G(t) = \hat{y}(t). \tag{9}$$

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The left-hand side is the total expenditure on consumption and saving. The right-hand side is the value of the total available income. The representative household has five variables, $s(t)$, $c_i(t)$, $c_a(t)$, $\hat{g}_G(t)$, and $l_h(t)$, to decide. The consumer's utility function is specified as follows

$$U(t) = \Gamma_c(E(t)) c_i^{\xi_0}(t) c_a^{\mu_0}(t) l_h^{\eta_0}(t) \hat{g}^{\gamma_0}(t) s^{\lambda_0}(t), \ \ \xi_0, \mu_0, \eta_0, \gamma_0, \lambda_0 > 0,$$

in which ξ_0, μ_0, η_0, γ_0, and λ_0 are the household's elasticity of utility with regard to industrial goods, agricultural goods, housing, use of gold, and saving. We call ξ_0, μ_0, η_0, γ_0, and λ_0 propensities to consume industrial goods, agricultural goods, housing, to use gold, and to hold wealth, respectively. Maximizing the utility function $U(t)$ subject to (9) yields

$$c_i(t) = \xi \hat{y}(t), \ \ p_a(t) c_a(t) = \mu \hat{y}(t), \ \ R_L(t) l_h(t) = \eta \hat{y}(t), \ \ R_G(t) \hat{g}(t) = \gamma \hat{y}(t), \ \ s(t) = \lambda \hat{y}(t), \tag{10}$$

where

$$\xi \equiv \frac{\rho \xi_0}{1 + \tilde{\tau}_c}, \ \ \mu \equiv \frac{\rho \mu_0}{1 + \tilde{\tau}_a}, \ \ \eta \equiv \frac{\rho \eta_0}{1 + \tilde{\tau}_h}, \ \ \gamma \equiv \frac{\rho \gamma_0}{1 + \tilde{\tau}_G}, \ \ \lambda \equiv \rho \lambda_0, \ \ \rho \equiv \frac{1}{\xi_0 + \mu_0 + \eta_0 + \gamma_0 + \lambda_0}.$$

Wealth accumulation

According to the definition of $s(t)$, the change in the household's wealth is given by

$$\dot{a}(t) = s(t) - a(t). \tag{11}$$

The equation simply states that the change in wealth is equal to saving minus dissaving.

Dynamics of pollutants and the environment sector

We now describe dynamics of the stock of pollutants, $E(t)$. We assume that pollutants are created both by production and consumption. We specify the dynamics of the stock of pollutants as follows

$$\dot{E}(t) = \theta_i F_i(t) + \theta_a F_a(t) + \tilde{\theta}_i C_i(t) + \tilde{\theta}_a C_a(t) - Q_e(t) - \theta_0 E(t), \tag{12}$$

in which θ_i, θ_a, $\tilde{\theta}_i$, $\tilde{\theta}_a$, and θ_0 are positive parameters and

$$Q_e(t) = A_e \Gamma_e(E) K_e^{\alpha_e}(t) N_e^{\beta_e}(t), \ \ A_e, \alpha_e, \beta_e > 0, \tag{13}$$

where $N_e(t)$ is the labor force employed by the environmental sector, A_e, α_e, and β_e are positive parameters, and $\Gamma_e(E)$ (≥ 0) is a function of E. The term $\theta_i F_i$ (or $\theta_a F_a$) means that pollutants that are emitted during production processes are linearly positively proportional to the output level

(Gutiérrez, 2008). The parameter, $\tilde{\theta}_i$ (or $\tilde{\theta}_a$) means that in consuming one unit of the good the quantity $\tilde{\theta}_i$ (or $\tilde{\theta}_a$) is left as waste (Prieur, 2009). The parameter $\tilde{\theta}_i$ depends on the technology and environmental sense of consumers. The parameter θ_0 is called the rate of natural purification. The term $\theta_0 E$ measures the rate that the nature purifies environment. The term, $K_e^{\alpha_e} N_e^{\beta_e}$, in Q_e means that the purification rate of environment is positively related to capital and labor inputs. The function, $\Gamma_e(E)$, implies that the purification efficiency is dependent on the stock of pollutants. For simplicity, we specify Γ_e as follows $\Gamma_e(E) = E^{b_e}$, $1 > b_e \ge 0$. This equation means that the productivity of the environment sector is positively related to the level of pollutants.

We now determine how the government determines the number of labor force and the level of capital employed for purifying pollution. We assume that all the tax incomes are spent on environment. The government's tax incomes consist of the tax incomes on the production sectors, consumption, wage income and wealth income. Hence, the government's income is given by

$$Y_e(t) = \tau_i F_i(t) + \tau_a p_a(t) F_a(t) + I_c(t) N, \tag{14}$$

where

$$I_c(t) \equiv \tilde{\tau}_c c_i(t) + \tilde{\tau}_a p_a(t) c_a(t) + \tilde{\tau}_h R(t) l_h(t) + \tau_k r(t) \bar{k}(t) + \tau_w w(t) + \tau_L R(t) \bar{l}(t) + \tau_G R_G(t) \bar{g}(t) + \tilde{\tau}_G R_G(t) \hat{g}_G(t).$$

The government budget is

$$(r(t) + \delta_k) K_e(t) + w(t) N_e(t) = Y_e(t). \tag{15}$$

The government maximizes (13) under constraint (15) as follows

$$\text{Max } Q_e(t) \quad \text{s.t. (15)}$$

The optimal solution is given by

$$(r(t) + \delta_k) K_e(t) = \alpha Y_e(t), \quad w(t) N_e(t) = \beta Y_e(t), \tag{16}$$

where

$$\alpha \equiv \frac{\alpha_e}{\alpha_e + \beta_e}, \quad \beta \equiv \frac{\beta_e}{\alpha_e + \beta_e}.$$

Balances of demand and supply for agricultural goods
The demand and supply for the agricultural sector's output balance at any point in time

$$C_a(t) = c_a(t)N = F_a(t). \tag{17}$$

All the land owned by households
The land owned by the population is equal to the national available land

$$\bar{l}(t)N = L. \tag{18}$$

Full employment of capital
We use $K(t)$ to stand for the total capital stock. We assume that the capital stock is fully employed. We have

$$K_i(t) + K_a(t) + K_e(t) = K(t). \tag{19}$$

where $K_e(t)$ is the capital stocks employed by the environmental sector.

Balances of demand and supply for industrial goods
The demand and supply for the industrial sector's output balance at any point in time

$$\dot{K}(t) = F_i(t) - c_i(t)N - \delta_k K(t). \tag{20}$$

According to Say's law, we can consider this equation redundant in the general equilibrium system.

The value of physical wealth and capital
The value of physical capital is equal to the value of physical wealth

$$\bar{k}(t)N = K(t). \tag{21}$$

The land and the gold owned by the households
The assets owned by the population are equal to the available amounts of the assets

$$\bar{l}(t)\bar{N} = L. \tag{22}$$
$$\bar{g}(t)\bar{N} = G. \tag{23}$$

Gold being fully used for decoration
We neglect possible holdings of gold by the government. The amount of gold used for decoration by the population is equal to the total gold

$$\hat{g}(t)N = G. \tag{24}$$

Full employment of labor force
We assume that labor force is fully employed

$$N_i(t) + N_a(t) + N_e(t) = N. \tag{25}$$

The land market clearing condition
The condition that land is fully used for the agricultural production and residential use implies

$$l_h(t)N + L_a(t) = L. \tag{26}$$

We thus built the model. We now examine dynamic properties of the model.

3. The dynamics and the motion by simulation

The previous section built the endogenous growth model of capital accumulation and environmental change with portfolio equilibrium between land, gold, and physical wealth. The model is based on microeconomic mechanisms to determine the prices, rents and distributions of land, gold, and physical wealth. The dynamic system contains many variables and these variables are interrelated nonlinearly. As it is almost impossible to give analytical solutions, we simulate the model. In the appendix, we show that the dynamics of the national economy can be expressed as two differential equations. First, we introduce a variable

$$z(t) \equiv \frac{r(t) + \delta_k}{w(t)}.$$

We now show that the dynamics can be expressed by two dimensional differential equations with $z(t)$ and $E(t)$ as the variables.

Lemma
The motion of the system is determined by the following two differential equations

$$\dot{z}(t) = \Lambda(z(t), E(t)),$$
$$\dot{E}(t) = \Omega(z(t), E(t)), \tag{27}$$

where the right-hand sides of (27) are functions of $z(t)$ and $E(t)$ determined in the appendix. Moreover, all the other variables can be determined as functions of $z(t)$ and $E(t)$ at any point in time by the following procedure: $r(t)$ and $w(t)$ by (A2) $\rightarrow \bar{k}_1(t)$ by (A31) $\rightarrow K_a(t)$ by (A18) $\rightarrow K_i(t)$ and

$K_e(t)$ by (A21) $\rightarrow$ $N_i(t)$, $N_a(t)$ and $N_e(t)$ by (A1) $\rightarrow \hat{y}(t)$ by (A14) $\rightarrow$ $\bar{l}$ by (13) $\rightarrow$ L_a and l_h by (A9) $\rightarrow$ $R_L(t)$ by (A15) $\rightarrow p_L(t)$ by (A21) $\rightarrow p_a(t)$ by (A5) $\rightarrow$ $\bar{g} = \hat{g}$ by (23) $\rightarrow R_G(t)$, $c_i(t)$, $c_a(t)$, and $s(t)$ by (10) $\rightarrow$ $p_G(t)$ by (A21) $\rightarrow a(t)$ by (A32) $\rightarrow F_i(t)$ by (1) $\rightarrow F_a(t)$ by (3) $\rightarrow Y_e(t)$ by (A28) $\rightarrow$ $Q_e(t)$ by (13).

The lemma shows that if we determine the values of the two variables with initial conditions, we determine all the other variables in the economic system. The lemma is important as it gives a procedure to follow the motion of the system with computer. The environment-related functions are specified as follows

$$\Gamma_m(t) = E^{-b_m}(t), \;\; m = i, a, e, c.$$

The parameter values are specified as follows

$$N = 20, \; L = 8, \; A_i = 1, \; A_a = 0.8, \; A_e = 0.5, \; \alpha_i = 0.32, \; \alpha_a = 0.1, \; \beta_a = 0.2, \; \alpha_e = 0.3,$$
$$\beta_e = 0.7, \; \lambda_0 = 0.7, \; \xi_0 = 0.15, \; \mu_0 = 0.06, \; \eta_0 = 0.06, \; b_i = b_a = 0.15, \; b_e = -0.1,$$
$$b_c = 0.1, \; \tau_m = 0.01, \; m = i, a, k, R, w, \; \tilde{\tau}_m = 0.01, \; m = h, a, c, \; \theta_i = 0.1, \; \theta_a = 0.05,$$
$$\tilde{\theta}_i = 0.05, \; \tilde{\theta}_a = 0.01, \; \theta_0 = 0.05, \; \delta_k = 0.05. \qquad (28)$$

The population is fixed at 20 and the land is 8. We assume that the propensity to save is much higher than the propensity to consume industrial goods and the propensity to consume agricultural goods. The tax rates are approximately 1 percent. The environmental impact parameters b_m are about 0.1 to 0.15. We specify the following initial conditions

$$z(0) = 1, \;\; E(0) = 16.$$

As shown in the appendix, the following variables are invariant in time

$$l_h = 0.236, \;\; L_a = 3.275, \;\; \bar{l} = 0.4, \;\; \bar{g} = \hat{g} = 0.05.$$

We plot the motion of the other variables in Figure 1. In Figure 1 the national gross product (GDP) is

$$Y(t) = F_i(t) + p_a(t) F_a(t) + l_h N R(t).$$

The GDP, national capital stock and total tax income fall over time till they become stationary. The wage rate, land price, price of agricultural goods, gold price, gold rent, and land rent are reduced. The rate of interest is enhanced. The output levels of the agricultural and industrial sectors are reduced. Some of the labor force is shifted from the agricultural and environment sectors to the industrial sector. The capital inputs of the three sectors are lowered. The physical wealth, total

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wealth, and consumption levels of the two goods are reduced. The output of the environment sector falls and environment is deteriorated.

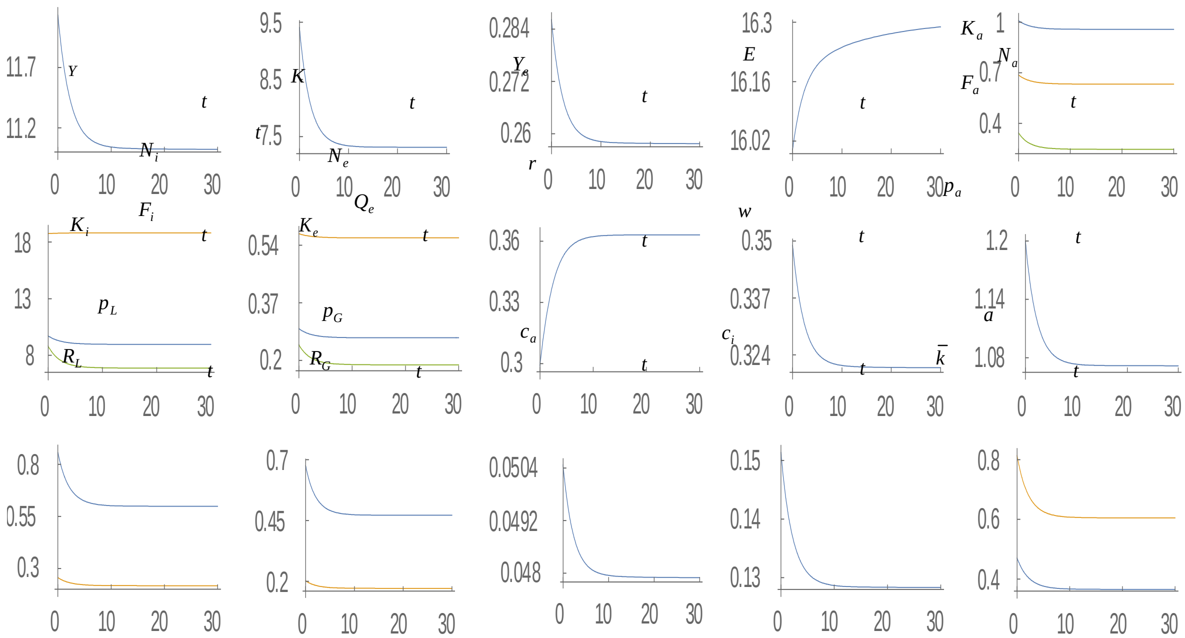

Figure 1 - The motion of the economic system

From Figure 1 we observe that all the variables tend to become stationary in the long term. This implies the existence of some equilibrium point. We confirm the existence of a unique equilibrium point as follows

$$Y = 11.02,\ Y_e = 0.26,\ K = 7.3,\ E = 16.3,\ w = 0.32,\ p_L = 0.6,\ R_L = 0.22,\ p_G = 0.47,$$
$$R_G = 0.17,\ r = 0.36,\ p_a = 1.07,\ F_a = 0.96,\ F_i = 8.97,\ Q_e = 0.67,\ K_a = 0.25,$$
$$K_i = 6.88,\ K_e = 0.19,\ N_a = 0.63,\ N_i = 18.8,\ N_e = 0.56,\ L_a = 3.28,\ \bar{l} = 0.4,$$
$$l_h = 0.24,\ \bar{g} = \hat{g} = 0.05,\ \bar{k} = 0.37,\ c_a = 0.05,\ c_i = 0.13,\ a = 0.61.$$

The two eigenvalues at the equilibrium point are

$$-0.319,\ -0.067.$$

The equilibrium point is stable. The stability is important as this guarantees that we can effectively conduct comparative dynamic analysis.

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Econ Res Guard 6(1): 18-44

4. Comparative dynamic analysis

We now conduct comparative dynamic analysis with regard some parameters. As the lemma provides a computational procedure to calibrate the motion of all the variables and the equilibrium point is locally stable, it is straightforward to follow the motion of the economic system. In the rest of this study we use $\overline{\Delta} x_j(t)$ to stand for the change rate of the variable, $x_j(t)$, in percentage due to changes in a parameter value.

A rise in the propensity to use gold

First we examine the effects of the following change in the propensity to use gold: $\gamma_0 : 0.01 \Rightarrow 0.015$. The land distribution and gold use per household are not affected

$$\overline{\Delta} L_a = \overline{\Delta} l_h = \overline{\Delta} \bar{l} = \hat{g} = 0.$$

The effects on the other variables are plotted in Figure 2. The prices and rents of land and gold are enhanced. The rate of interest falls and the wage rate rises. The price of agricultural goods is increased. The household's physical wealth and total wealth are augmented. The GDP, national capital, and total tax income are augmented. The environment is deteriorated. The household consumes more agricultural and initial goods. The output levels and capital inputs of the three sectors are enhance. The labor input of the industrial sector is reduced and the other two sectors' labor inputs are augmented.

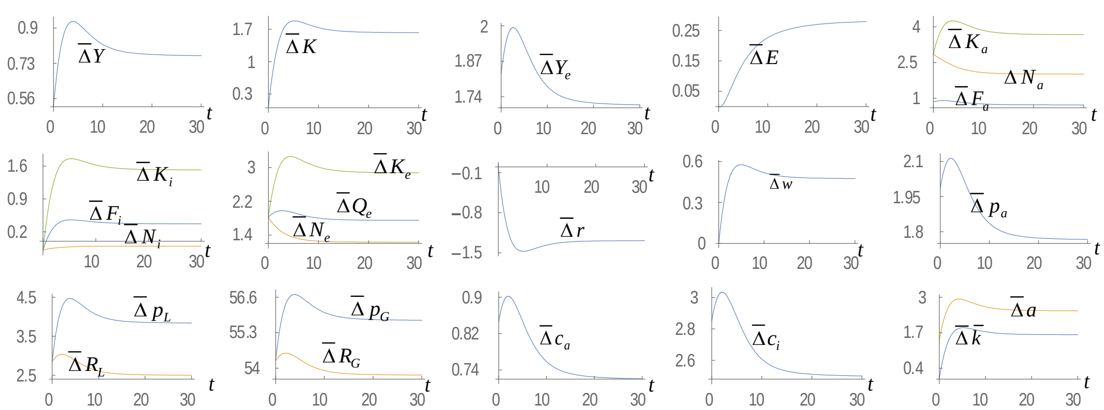

Figure 2 - A rise in the propensity to use gold

The propensity to consume housing being increased

We now allow the following change in the propensity to consume housing: $\eta_0 : 0.06 \Rightarrow 0.07$. The rise in the household's preference for housing lead to the following changes in the land-use distribution

$$\overline{\Delta} L_a = -8.96, \ \overline{\Delta} l_h = 6.21, \ \overline{\Delta} \bar{l} = \overline{\Delta} \hat{g} = 0.$$

Less land is used for agricultural production and more land is for the residential use. The effects on the other variables are plotted in Figure 3. The land value rises initially and falls in the long term. The land rent, gold rent and gold price are enhanced. The rate of interest is increased. The wage rate falls. The price of agricultural goods is increased. The household's physical wealth is reduced. The household's total wealth rise initially and falls in the long term. The consumption levels of the two goods fall. The national capital stock is reduced. The GDP and total income rise initially and fall in the long term. The environment is improved. The output levels of the industrial and agricultural sectors fall. The environmental sector's output and capital input rise initially and fall in the long term. The labor input of the agricultural sector falls and the labor inputs of the other two sectors rise.

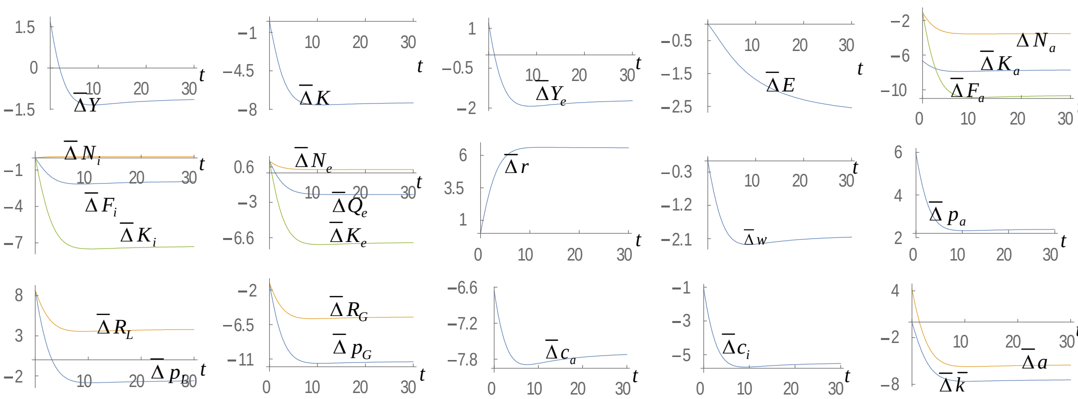

Figure 3 - The propensity to consume housing being increased

The propensity to consume industrial goods being enhanced

Suppose that the propensity to consume industrial goods is increased as follows: $\xi_0 : 0.15 \Rightarrow 0.16$. The land distribution and gold amount per household are not affected. The effects on the other variables are plotted in Figure 4. The total capital stock and the GDP are reduced. The environment is deteriorated. The rate of interest rises and the wage rate falls. The prices and rents of gold and land are reduced. The total wealth and physical wealth per household are reduced. The household's consumption level of industrial goods rises. The household's consumption level of agricultural goods falls. The output level and the two inputs of the environment sector are increased. The price of agricultural goods falls. The labor force is shifted from the agricultural and environment sectors to the industrial sector. The capital inputs of all the sectors are reduced in the long term. The government's income rises initially and falls in the long term.

t $\bar{\Delta} F_a$

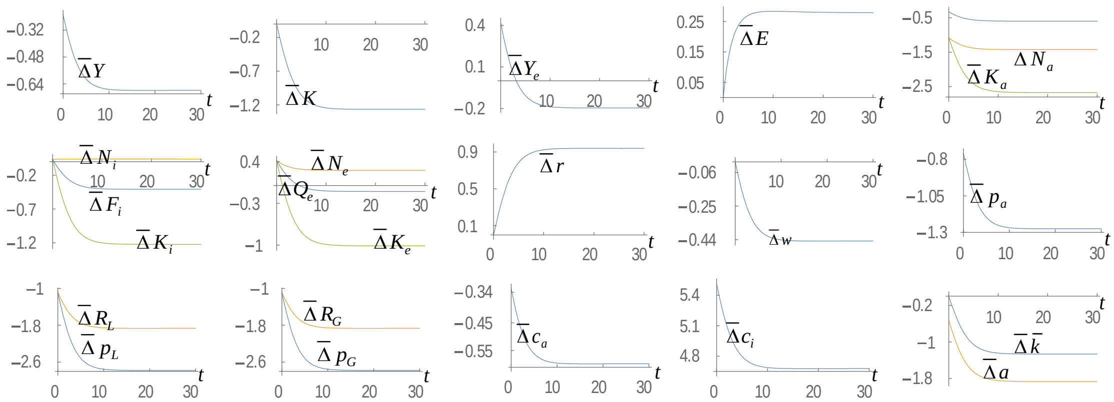

Figure 4 - The propensity to consume industrial goods being enhanced

The propensity to consume agricultural goods being enhanced

We now study the effects that the propensity to consume agricultural goods is increased as follows: $\mu_0 : 0.06 \Rightarrow 0.07$. The land is reallocated as follows

$$\bar{\Delta} L_a = 9.22, \ \bar{\Delta} l_h = -6.39, \ \bar{\Delta} \bar{l} = \bar{\Delta} \hat{g} = 0.$$

The gold amount per household is not changed. Some of the land is shifted to the residential use to the agricultural use. The effects on the other variables are plotted in Figure 5. The output level and capital and labor inputs of the agricultural sector are increased. The total capital stock is reduced. The GDP and the total tax income rise initially and fall in the long term. The wage rate falls and the rate of interest rises. The gold rent and price fall. The household holds less physical wealth. The household's total wealth rises initially and falls in the long term. The household's consumption level of agricultural goods is increased and consumption level of industrial goods is reduced. The price of agricultural goods rises. The output level and two input factors of the agricultural sector are augmented. The output level and two input factors of the industrial sector are reduced. The government's income rises initially and falls in the long term. The environment is enhanced.

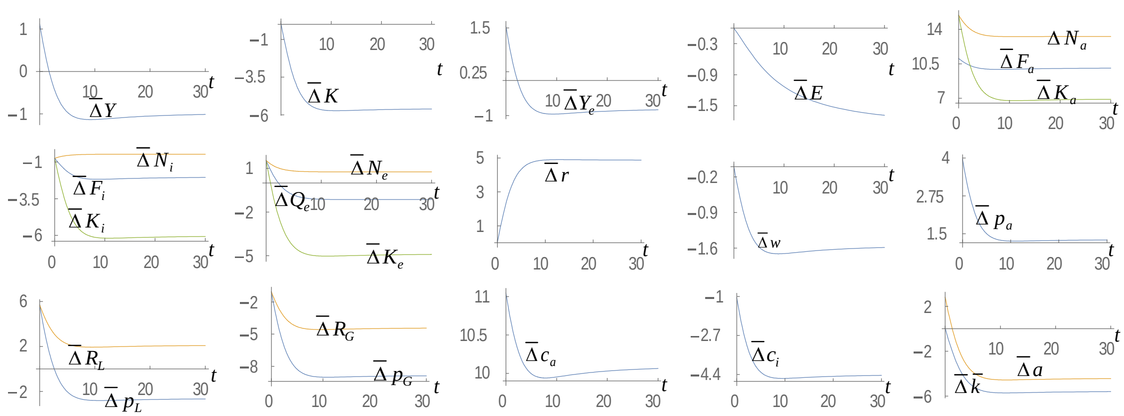

Figure 5 - The propensity to consume agricultural goods being enhanced

The propensity to save being enhanced

We now allow the propensity to save to rise as follows: $\lambda_0 : 0.7 \Rightarrow 0.71$. There is no impact on land-use allocation and gold amount per household. The effects are plotted in Figure 6. The household has more physical wealth and the nation holds more physical capital. The household's wealth falls initially and rises in the long term. The GDP is reduced initially and augmented in the long term. The government's tax income rises initially and falls in the long term. The output level and two inputs of the environment sector fall initially and rise in the long term. The environment is deteriorated. The wage rate is enhanced and the rate of interest is reduced. The rents of land and gold are reduced. The prices of land and gold fall initially and rise in the long term. The consumption levels of the two goods are reduced. The total wealth falls initially and rises in the long term. The output level and two inputs of the industrial sector are augmented. The labor force is shifted from the agricultural and environment sectors to the industrial sectors.

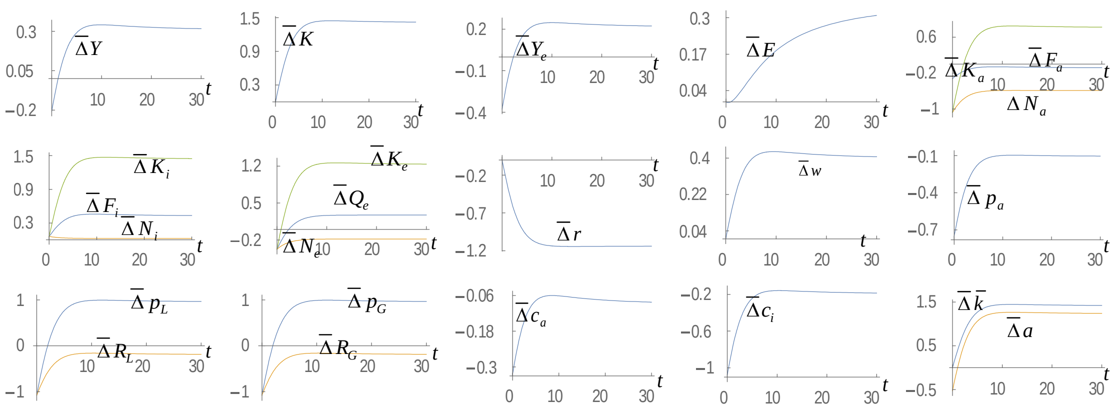

Figure 6 - The propensity to save being enhanced

The environment sector's productivity being enhanced

We now allow the environment sector's productivity to rise as follows: $A_e : 0.5 \Rightarrow 0.6$. There is no impact on land-use allocation and gold amount per household. The effects are plotted in Figure 6. The output level and two inputs of the environment sector are enhanced. The total tax income is increased. The environmental quality is improved. The labor distribution is slightly affected. The output levels and capital inputs of the agricultural and industrial sectors are increased. The household has more physical wealth and total wealth. The nation holds more physical capital. The GDP is augmented. The wage rate and the rate of interest rise. The rents and prices of land and gold are enhanced. The price of agricultural goods is reduced initially and increased in the long term. The household consumes more two goods.

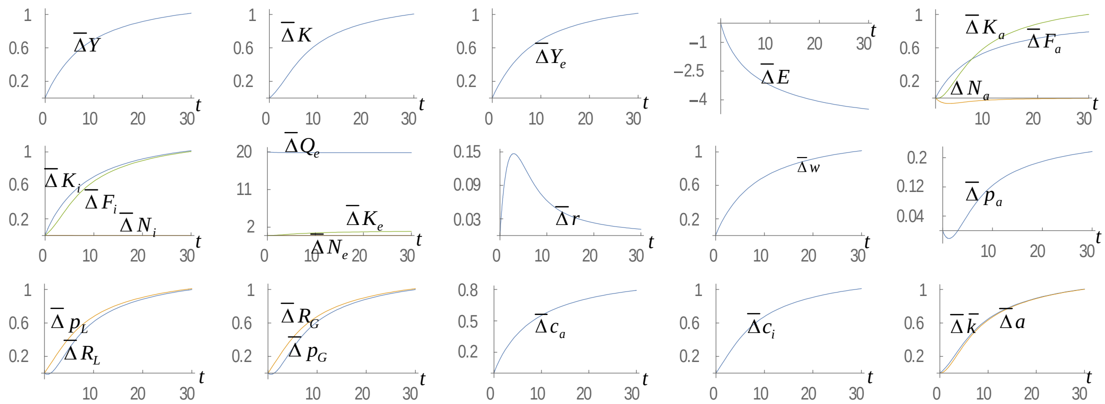

Figure 7 - The environment sector's productivity being enhanced

The tax rate on consumption of industrial goods being increased

We now allow the tax rate on consumption of industrial goods to be increased as follows: $\tilde{\tau}_c : 0.01 \Rightarrow 0.015$. The land use pattern and gold amount per household are not affected. The effects on the other variables are plotted in Figure 8. The consumption of industrial goods is reduced and the consumption of agricultural goods is enhanced. The total capital stock and GDP are slightly augmented. The wage rate, the rate of interest, and the rents and prices of land and gold are slightly increased. The total wealth and physical wealth per household are increased. The price of agricultural goods falls initially and rises in the long term. The environmental quality is enhanced.

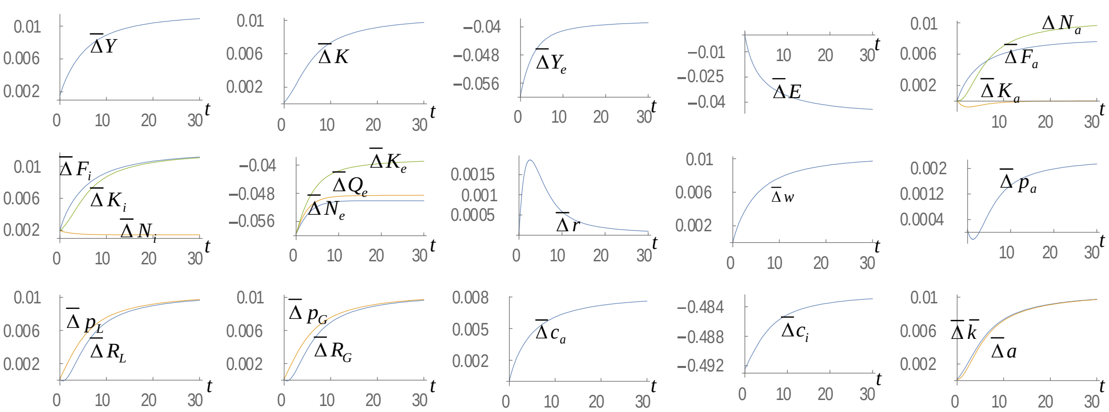

Figure 8 - The tax rate on consumption of industrial goods being increased

A rise in the population

We now study the effects of the following population increase: $N: 20 \Rightarrow 21$. The impact on the land use and gold amount per household are as follows

$$\overline{\Delta} L_a = 0, \ \overline{\Delta} l_h = \overline{\Delta} \bar{l} = \overline{\Delta} \hat{g} = -4.76.$$

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Econ Res Guard 6(1): 18-44

The agricultural land use for agriculture is not affected. The lot size and per household's land are reduced. The effects on the other variables are plotted in Figure 9. The GDP and national physical capital are augmented. The total tax income rises. The environmental quality is deteriorated. The output levels and inputs of the three sectors are increased. The rents and prices of land and gold are increased. The household owns less physical wealth and total wealth and consumes less agricultural and industrial goods.

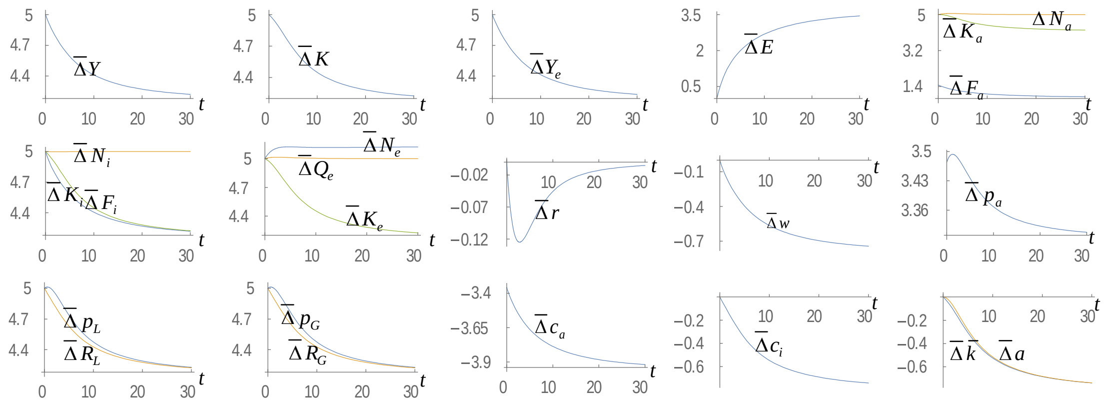

Figure 9 - A Rise in the population

The tax rate on the land rent income being augmented

We now allow the tax rate on land ownership to be increased as follows: $\tau_R : 0.01 \Rightarrow 0.015$. The land-use pattern is not affected. The effects on the other variables are plotted in Figure 10. The land rent rises. The land price falls initially but rises in the long term. The gold rent and price are enhanced. The rate of interest falls and wage rate rises. The price of the agricultural good is increased. The physical wealth and the consumption levels of the two goods are increased. The total wealth per household and GDP fall initially and rise in the long term. The national physical capital and total tax income are augmented. The environmental quality is improved. The output levels of the three sectors are all increased.

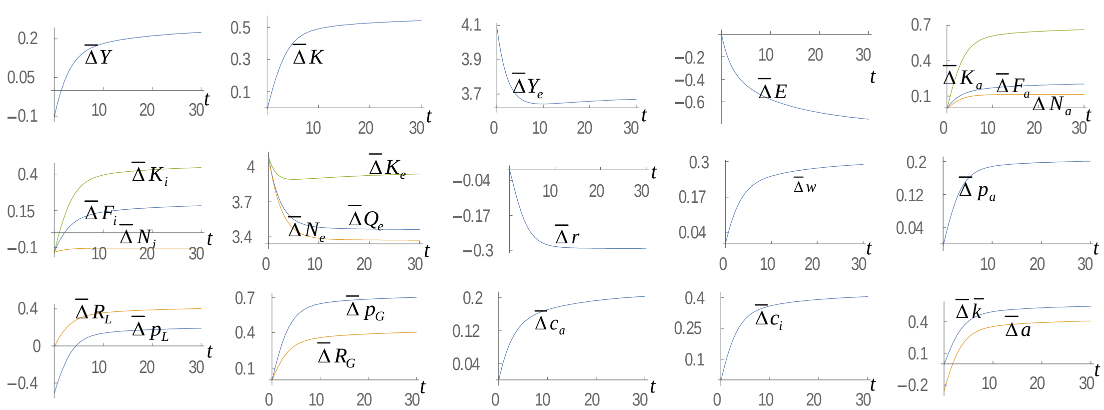

Figure 10 - The tax rate on the land rent income being augmented

5. Concluding remarks

This study built an endogenous growth model of capital accumulation and environmental change with portfolio equilibrium between land, gold, and physical wealth. The model is based on microeconomic mechanisms for determining the prices, rents and distributions of land, gold, and physical wealth. The economy consists of industrial, agricultural, and environmental sectors. The land is distributed between agricultural production and residential use. Gold is used for decoration and owned for storing value. The model is an integration of the neoclassical growth theory and Ricardian theory. We simulated the motion of the economy. We demonstrated that the economic system has a unique stable steady state. We also conducted comparative dynamic analysis to demonstrate how exogenous changes in the propensity to use gold, the propensity to consume housing, the propensity to consume industrial goods, the propensity to consume agricultural goods, the propensity to save, the environment sector's productivity, the tax rate on consumption of industrial goods, the population, and the tax rate on the land rent income affect the economic system during transitory processes and in long-term equilibrium. The theoretical model is obtained under some strict conditions. For instance, we don't take account of many possible important determinants of land and gold prices. Many limitations of this model become apparent in the light of the sophistication of the literature of growth theory, land economics and portfolio choice theory. The model can be extended and generalized in different directions. For instance, it is important to study the economic dynamics when utility and production functions are taken on other functional forms.

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Appendix: Proving the lemma

The appendix shows that the dynamics can be expressed by two differential equations. From (2), (4), and (16) we obtain

$$z \equiv \frac{r + \delta_k}{w} = \frac{\tilde{\alpha}_i N_i}{K_i} = \frac{\tilde{\alpha}_a N_a}{K_a} = \frac{\tilde{\alpha}_e N_e}{K_e}, \tag{A1}$$

where $\tilde{\alpha}_j \equiv \alpha_j / \beta_j$, $j = i, a, e$. By (1) and (2), we have

$$r + \delta_k = \frac{\bar{\tau}_i \alpha_i A_i}{\tilde{\alpha}_i^{\beta_i}} \Gamma_i z^{\beta_i}, \quad w = \frac{\bar{\tau}_i \tilde{\alpha}_i^{\alpha_i} \beta_i A_i \Gamma_i}{z^{\alpha_i}}, \tag{A2}$$

where we also use (A1). We express w and r as functions of z and E.

From (10) and (17), we get

$$\mu \hat{y} N = p_a F_a. \tag{A3}$$

From (4), we have

$$r + \delta_k = \frac{\bar{\tau}_a \alpha_a p_a F_a}{K_a}. \tag{A4}$$

From (A4) and (3) we solve

$$p_a \left(\frac{L_a}{K_a} \right)^{\varsigma} = \frac{\tilde{\alpha}_a^{\beta_a} (r + \delta_k)}{\bar{\tau}_a \alpha_a A_a \Gamma_a z^{\beta_a}}, \tag{A5}$$

where we use (A1). From (A3) and (A4), we solve

$$\mu \hat{y} N = \left(\frac{r + \delta_k}{\bar{\tau}_a \alpha_a} \right) K_a. \tag{A6}$$

By (4) and (A3), we have

$$R_L = \frac{\bar{\tau}_a \varsigma \mu \hat{y} N}{L_a}. \tag{A7}$$

From $R l_h = \eta \hat{y}$ in (10) and (A7), we have

$$\bar{\tau}_a \varsigma \mu N l_h = \eta L_a. \tag{A8}$$

From (26) and (A8), we solve the land distribution as follows

$$L_a = \frac{\bar{\tau}_a \varsigma \mu L}{\eta + \bar{\tau}_a \varsigma \mu}, \quad l_h = \frac{\eta L}{(\eta + \bar{\tau}_a \varsigma \mu) N}. \tag{A9}$$

The land distribution is invariant over time.

From the definition of $\hat{y}$, we have

$$\hat{y} = (1 + (1 - \tau_k) r)\bar{k} + (1 - \tau_w) w + (1 - \tau_L)\frac{R L}{N} + p_L \frac{L}{N} + (1 - \tau_G)\frac{R_G G}{N} + p_G \frac{G}{N}, \tag{A10}$$

where we also use (22) and (23). Insert (5) in (A10)

$$\hat{y} = r_k \bar{k} + (1 - \tau_w) w + \frac{r_L}{N}\left[(1 - \tau_L) L R_L + (1 - \tau_G) G R_G\right], \tag{A11}$$

where

$$r_k(z, E) \equiv 1 + (1 - \tau_k) r, \quad r_L(z, E) \equiv 1 + \frac{1}{(1 - \tau_k) r}.$$

Insert (24) in (10)

$$R_G = \frac{\gamma N \hat{y}}{G}. \tag{A12}$$

Insert (A7) and (A12) in (A11)

$$\hat{y} = r_k \bar{k} + (1 - \tau_w) w + r_L\left[(1 - \tau_L) L \frac{\bar{\tau}_a \varsigma \mu \hat{y}}{L_a} + (1 - \tau_G) \gamma \hat{y}\right], \tag{A13}$$

Solve (A13)

$$\hat{y} = \tilde{\omega}_1 \bar{k} + \tilde{\omega}_2, \tag{A14}$$

where

$$\tilde{\omega}_1 \equiv r_k \tilde{\omega}_0 \quad \tilde{\omega}_2 \equiv (1-\tau_w) w \tilde{\omega}_0, \quad \tilde{\omega}_0 \equiv \left\{1 - r_L \left[(1-\tau_L) L \frac{\bar{\tau}_a \varsigma \mu}{L_a} + (1-\tau_G)\gamma\right]\right\}^{-1}.$$

From $R_L l_h = \eta \hat{y}$ in (10) and (A14) we solve

$$R_L = \omega_1 \bar{k} + \omega_2, \tag{A15}$$

where

$$\omega_1 \equiv \frac{\eta \tilde{\omega}_1}{l_h}, \quad \omega_2 \equiv \frac{\eta \tilde{\omega}_2}{l_h}.$$

Insert (A1) in $N_i + N_a + N_e = N$

$$\frac{K_i}{\tilde{\alpha}_i} + \frac{K_a}{\tilde{\alpha}_a} + \frac{K_e}{\tilde{\alpha}_e} = \frac{N}{z}. \tag{A16}$$

From (19) and (21) we have

$$K_i + K_a + K_e = N \bar{k}. \tag{A17}$$

From (A6) and (A14) we solve

$$K_a = \hat{\omega}_1 \bar{k} + \hat{\omega}_2, \tag{A18}$$

where

$$\hat{\omega}_1(z, E) \equiv \tilde{\omega}_1 \mu N \left(\frac{\alpha_a \bar{\tau}_a}{r + \delta_k}\right), \quad \hat{\omega}_2(z, E) \equiv \tilde{\omega}_2 \mu N \left(\frac{\alpha_a \bar{\tau}_a}{r + \delta_k}\right).$$

Insert (A18) in, respectively, (A16) and (A17)

$$\begin{aligned} &\frac{K_i}{\tilde{\alpha}_i} + \frac{K_e}{\tilde{\alpha}_e} = b_1(z, E) \equiv \frac{N}{z} - \frac{\hat{\omega}_2}{\tilde{\alpha}_a} - \frac{\hat{\omega}_1 \bar{k}}{\tilde{\alpha}_a}, \\ &K_i + K_e = b_2(z, E) \equiv N \bar{k} - \hat{\omega}_1 \bar{k} - \hat{\omega}_2. \end{aligned} \tag{A19}$$

Solve (A19)

$$K_i = \alpha_0 b_1 - \frac{\alpha_0 b_2}{\tilde{\alpha}_e}, \quad K_e = \frac{\alpha_0 b_2}{\tilde{\alpha}_i} - \alpha_0 b_1, \tag{A20}$$

where

$$\alpha_0 \equiv \left(\frac{1}{\tilde{\alpha}_i} - \frac{1}{\tilde{\alpha}_e} \right)^{-1}.$$

Insert the definitions of b_j in (A20)

$$K_i = m_i \bar{k} - \bar{m}_i, \quad K_e = m_e \bar{k} - \bar{m}_e, \tag{A21}$$

where

$$m_i(z, E) \equiv -\frac{\alpha_0 \hat{\omega}_1}{\tilde{\alpha}_a} - \frac{\alpha_0}{\tilde{\alpha}_e} N + \frac{\alpha_0}{\tilde{\alpha}_e} \hat{\omega}_1, \quad \bar{m}_i(z, E) \equiv \frac{\alpha_0 \hat{\omega}_2}{\tilde{\alpha}_a} - \frac{\alpha_0 N}{z} - \frac{\alpha_0}{\tilde{\alpha}_e} \hat{\omega}_2,$$
$$m_e(z, E) \equiv \frac{\alpha_0 N}{\tilde{\alpha}_i} - \frac{\alpha_0 \hat{\omega}_1}{\tilde{\alpha}_i} + \frac{\alpha_0 \hat{\omega}_1}{\tilde{\alpha}_a}, \quad \bar{m}_e(z, E) \equiv \frac{\alpha_0 \hat{\omega}_2}{\tilde{\alpha}_i} + \frac{\alpha_0 N}{z} - \frac{\alpha_0 \hat{\omega}_2}{\tilde{\alpha}_a}.$$

By (A18) and (A21), we solve the capital distribution as functions of z, E and $\bar{k}$. By (A1), we solve the labor distribution as functions of z, E and $\bar{k}$ as follows

$$N_i = \frac{z K_i}{\tilde{\alpha}_i}, \quad N_a = \frac{z K_a}{\tilde{\alpha}_a}, \quad N_e = \frac{z K_e}{\tilde{\alpha}_e}. \tag{A22}$$

From (5)

$$p_L = \frac{(1 - \tau_R) R_L}{(1 - \tau_k) r}, \quad p_G = \frac{(1 - \tau_G) R_G}{(1 - \tau_k) r}. \tag{A23}$$

Insert (2) and (A18) in (20)

$$Y_e = \left(\frac{r + \delta_k}{\alpha_i \bar{\tau}_i} \right) \tau_i K_i + \left(\frac{r + \delta_k}{\alpha_a \bar{\tau}_a} \right) \tau_a K_a + I_c N. \tag{A24}$$

Substituting (A16) and (A19) into (A22) yields

$$Y_e = I_k \bar{k} - I_0 + I_c N, \tag{A25}$$

where

$$I_k(z, E) \equiv \left(\frac{r + \delta_k}{\alpha_i \bar{\tau}_i}\right) \tau_i m_i + \left(\frac{r + \delta_k}{\alpha_a \bar{\tau}_a}\right) \tau_a \hat{\omega}_1, \quad I_0(z, E) \equiv \left(\frac{r + \delta_k}{\alpha_i \bar{\tau}_i}\right) \tau_i \bar{m}_i - \left(\frac{r + \delta_k}{\alpha_a \bar{\tau}_a}\right) \tau_a \hat{\omega}_2 .$$

Insert (10) in the definition of I_c

$$I_c = \tau_0 \hat{y} + \tau_k r \bar{k} + \tau_w w, \tag{A26}$$

where

$$\tau_0 \equiv \tilde{\tau}_c \xi + \tilde{\tau}_a \mu + \tilde{\tau}_h \eta + \frac{\eta \tau_R \bar{l}}{l_h} + \tau_G \gamma + \tilde{\tau}_G \gamma .$$

Insert (A14) in (A26)

$$I_c = (\tau_0 \tilde{\omega}_1 + \tau_k r)\bar{k} + \tau_0 \tilde{\omega}_2 + \tau_w w. \tag{A27}$$

Insert (A27) in (A25)

$$Y_e = \tilde{m}_0 \bar{k} - I_0 + \tau_0 \tilde{\omega}_2 N + \tau_w w N, \tag{A28}$$

where

$$\tilde{m}_0 \equiv I_k + \tau_0 \tilde{\omega}_1 N + \tau_k r N .$$

From (16) and (A28) we solve

$$m_0 K_e = \tilde{m}_0 \bar{k} - I_0 + \tau_0 \tilde{\omega}_2 N + \tau_w w N, \tag{A29}$$

where $m_0 \equiv (r + \delta_k)/\alpha$. Insert (A21) in (A29)

$$m_0 m_e \bar{k} - m_0 \bar{m}_e = \tilde{m}_0 \bar{k} - I_0 + \tau_0 \tilde{\omega}_2 N + \tau_w w N, \tag{A30}$$

From (A30) we solve

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Econ Res Guard 6(1): 18-44

$$\bar{k}(z, E) = \left(m_0\, \overline{m}_e - I_0 + \tau_0\, \tilde{\omega}_2\, N + \tau_w\, w\, N\right)\left(m_0\, m_e - \tilde{m}_0\right)^{-1}. \tag{A31}$$

From (6) and (A31) we have

$$a = \phi(z, E) \equiv \bar{k} + \frac{(1-\tau_L)\bar{l}\, R_L}{(1-\tau_k)r} + \frac{(1-\tau_G)G\, R_G}{(1-\tau_k)r\, N}. \tag{A32}$$

It is straightforward to check that all the variables can be expressed as functions of z and E at any point in time as follows: r and w by (A2) $\rightarrow \bar{k}$ by (A31) $\rightarrow K_a$ by (A18) $\rightarrow K_i$ and K_e by (A21) $\rightarrow$ N_i, N_e, and N_a by (A1) $\rightarrow \hat{y}$ by (A14) $\rightarrow \bar{l}$ by (22) $\rightarrow L_a$ and l_h by (A9) $\rightarrow R_L$ by (A15) $\rightarrow p_L$ by (A21) $\rightarrow p_a$ by (A5) $\rightarrow \bar{g} = \hat{g}$ by (23) $\rightarrow R_G$, c_i, c_a, and s by (10) $\rightarrow p_G$ by (A21) $\rightarrow a$ by (A32) $\rightarrow F_i$ by (1) $\rightarrow F_a$ by (3) $\rightarrow Y_e$ by (A28) $\rightarrow Q_e$ by (13). From this procedure, (18) and (11), we have

$$\dot{a} = \Lambda_0(z, E) \equiv s - a, \tag{A33}$$

$$\dot{E} = \Omega(z, E) \equiv \theta_i\, F_i + \theta_a\, F_a + \tilde{\theta}_i\, C_i + \tilde{\theta}_a\, C_a - Q_e - \theta_0\, E. \tag{A34}$$

Taking derivatives of (A32) with respect to t yields

$$\dot{a} = \frac{\partial \phi}{\partial z}\dot{z} + \Omega \frac{\partial \phi}{\partial E}, \tag{A35}$$

in which we use (A34). Equal (A33) and (A35)

$$\dot{z} = \Lambda(z, E) \equiv \left(\Lambda_0 - \Omega \frac{\partial \phi}{\partial E}\right)\left(\frac{\partial \phi}{\partial z}\right)^{-1}. \tag{A36}$$

From (A34) and (A36), we determine the motion of z and E. We thus proved the lemma.